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On a conjecture of E. Rapaport Strasser about knot-like groups and its pro-p version

Dessislava H. Kochloukova[1]
UNICAMP-IMECC, Cx. P. 6065, 13083-970 Campinas, SP, Brazil

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Abstract

A group G is knot-like if it is finitely presented of deficiency 1 and has abelianization $G/G' \simeq \mathbb{Z}$. We prove the conjecture of E. Rapaport Strasser that if a knot-like group G has a finitely generated commutator subgroup G' then G' should be free in the special case when the commutator G' is residually finite. It is a corollary of a much more general result : if G is a discrete group of geometric dimension n with a finite $K(G,1)$-complex Y of dimension n, Y has Euler characteristics 0, N is a normal residually finite subgroup of G, N is of homological type FP_{n-1} and $G/N \simeq \mathbb{Z}$ then N is of homological type FP_n and hence G/N has finite virtual cohomological dimension $vcd(G/N) = cd(G) - cd(N)$. In particular either N has finite index in G or $cd(N) \leqslant cd(G) - 1$.

Furthermore we show a pro-p version of the above result with the weaker assumption that G/N is a pro-p group of finite rank. Consequently a pro-p version of Rapaport's conjecture holds.

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0. Introduction

0.1. Introduction to Rapaport's conjecture

A finitely presented group is called knot-like if it has abelianization $\mathbb{Z}$ and deficiency 1 [23]. By definition the deficiency of a group G is the maximum of $n - r$ over all finite

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E-mail address: desi@ime.unicamp.br.

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presentations of G, where n is the number of generators and r is the number of relations. It was conjectured by E. Rapaport Strasser that if the commutator group G' of a knot-like group G is finitely generated then G' should be free, and established in the case when G is 2-generator, 1 relator group [23]. In this paper we prove the conjecture when G' is residually finite using some results of [12] about groups of deficiency 1 and some results about higher dimensional homological Σ-invariants of groups [4,5]. The paper [12] uses substantially L^2-homology methods i.e. the L^2-Betti numbers of spaces and groups but here we just apply already existing results without exploring further the L^2-homology technique. Note that by [12, Theorem 2 + Lemma 2] it follows that a knot-like group G with finitely generated commutator is always of geometric dimension at most 2, hence is of cohomological dimension at most 2.

By a result of Bieri if N is a normal subgroup in a group G of cohomological dimension $cd(G) = 2$ and N is of homological type FP_2 then either N is free or N has a finite index in G [2, Theorem B + Remark 5.4]. Then as observed in [12, Corollary, p. 273] to prove Rapaport's conjecture we are left to show that the commutator G' is of type FP_2. Luckily there is a whole theory due to Bieri and Renz [4] that studies when the commutator of a group H of homological type FP_m is of the same homological type FP_m by the means of a homological invariant $\Sigma^m(H, \mathbb{Z})$. A new approach to the low dimensional case $m = 1$ of the Σ^m-invariant was suggested in [29] where geometric methods are applied. Later on this approach was generalised in [5] to show that for a (left) $\mathbb{Z}[G]$-module A of homological type FP_m we have $[\chi] \in \Sigma^m(G, A)$ if and only if $Tor_i^{\mathbb{Z}[G]}(\widehat{\mathbb{Z}G}_\chi, A) = 0$ for all $i = 0, 1, \dots, m$, where $\widehat{\mathbb{Z}G}_\chi$ is the Novikov completion of $\mathbb{Z}[G]$ at χ.

0.2. The main results in the discrete case

By definition a module A over a ring R is of homological type FP_m if there is a projective resolution of A with all modules in dimension at most m being finitely generated. A group G is of type FP_m if the trivial module $\mathbb{Z}$ is of homological type FP_m over the group ring $\mathbb{Z}[G]$. A group G is said to be of geometric dimension at most n if there is a $K(G, 1)$ CW-complex Y of dimension n i.e. Y is a connected CW-complex of dimension n with fundamental group isomorphic to G and $\pi_i(Y) = 1$ for all $i \geqslant 2$. The geometric dimension of G is the minimal possible dimension for such Y. If we assume further that there is a finite $K(G, 1)$ CW-complex Y of finite dimension, by definition the geometric Euler characteristics of G is the Euler characteristics of the complex Y i.e. the alternating sum $\sum_{i \geqslant 0} (-1)^i m_i$, where m_i is the number of cells of Y of dimension i. Note that the geometric Euler characteristics coincides with the Euler characteristics for torsion-free groups of finite homological type as defined in [6, p. 247]. The following theorem is one of the main results of this paper.

Theorem 1. *Let G be a non-trivial discrete group of geometric dimension n with a finite $K(G, 1)$ CW-complex Y of dimension n such that the Euler characteristics of Y is zero i.e. the geometric Euler characteristics of G is zero. Suppose that N is a residually finite normal subgroup of G such that N is of homological type FP_{n-1} and $G/N \simeq \mathbb{Z}$. Then N is of homological type FP_n.*

Note that the condition on the geometric Euler characteristics of G cannot be removed as the following example from [2] shows. Let $G = F_2 \times F_2 = \langle u, v\rangle \times \langle x, y\rangle$ be the direct product of two free groups of rank two. Let N be the kernel of the homomorphism of G to $\mathbb{Z}$ sending u, y to 0 and v, x to 1. Then N is finitely generated by u, y, vx^{-1} but is not free, and hence by [2, Theorem B + Remark 5.4] is not of homological type FP_2. The group G has geometric dimension 2, there is an obvious finite $K(G, 1)$-complex with one vertex, four edges and four 2-cells, hence the geometric Euler characteristics of G is not 0.

The above example generalises to the free product G_n of n copies of F_2. By the main result of [18] the homological invariants up to dimension n form a strict chain $\Sigma^n(G_n, \mathbb{Z}) \subset \Sigma^{n-1}(G_n, \mathbb{Z}) \subset \cdots \subset \Sigma^1(G_n, \mathbb{Z})$, hence G_n has a subgroup H containing the commutator subgroup of G_n such that H is of type FP_{n-1} but not of type FP_n. Note that G_n has geometric dimension n but its Euler characteristics is not 0.

We discuss several corollaries of Theorem 1. One of them, Corollary 2, proves the Rapaport's conjecture when G' is residually finite.

Corollary 1. *Let G be a non-trivial discrete group of geometric dimension n with a finite $K(G, 1)$ CW-complex Y of dimension n such that the Euler characteristics of Y is zero i.e. the geometric Euler characteristics of G is zero. Suppose that N is a residually finite normal subgroup of G such that N is of homological type FP_{n-1} and $G/N \simeq \mathbb{Z}$. Then G/N has finite virtual cohomological dimension*

$$vcd(G/N) = cd(G) - cd(N).$$

In particular either N has finite index in G or N has cohomological dimension at most $cd(G) - 1$.

Corollary 2. *Let G be a knot-like group with finitely generated, residually finite commutator subgroup G'. Then G' is free.*

Finally we obtain some results that imply that the Euler characteristics is zero. Let X be a finite aspherical polyhedron. By [11] if $\Gamma = \pi_1(X)$ has a non-trivial centre then the Euler characteristics $\chi(X) = 0$. This result was extended in [25] to the case when Γ has non-trivial normal abelian subgroup. Here we show a similar result holds for a different class of groups.

Theorem 2. *Let X be a finite aspherical polyhedron and let $G = \pi_1(X)$. If $\Sigma^\infty(G, \mathbb{Z}) = \bigcap_{i \geqslant 1} \Sigma^i(G, \mathbb{Z})$ is non-empty then $\chi(X) = 0$.*

We observe that the condition that $\Sigma^\infty(G, \mathbb{Z})$ is non-empty is quite strong. It is equivalent to the existence of a non-zero character $\chi : G \to \mathbb{R}$ such that $\mathbb{Z}$ has homological type FP_∞ as a left $\mathbb{Z}[G_\chi]$-module. Though the group G from Theorem 2 is of type FP_∞ in general this property does not pass to submonoids. For example, for a finitely generated free group F even the first invariant $\Sigma^1(F, \mathbb{Z})$ is empty.

Corollary 3. *Let X be a finite aspherical polyhedron and let $G = \pi_1(X)$. If G has a subgroup N of infinite index in G, N is of type FP_∞ and contains the commutator of G then $\chi(X) = 0$.*

It will be interesting to find out whether there are groups satisfying Corollary 3 that are not soluble torsion-free constructible groups.

0.3. A result about the Kaplansky property

We say that a ring R with $1 \neq 0$ has the Kaplansky property if for any $n \geqslant 1$ and any two matrices $A, B \in M_n(R)$ if AB is the identity matrix I_n then $BA = I_n$. It is known that for a field k of characteristics zero and an arbitrary group G the group algebra $k[G]$ has the Kaplansky property [14, p. 122, 19,21, p. 38, Chapter 2.1] but it is still an open problem whether this holds in positive characteristic, though some cases were recently proved in [1]. In order to establish the results from the previous section we will need the following theorem.

Theorem 3. *Let G be a finitely generated group, $\chi : G \to \mathbb{R}$ a non-zero character and N a finitely generated, residually finite subgroup of $Ker(\chi)$ such that $G' \subseteq N$. Then the Novikov ring $\widehat{\mathbb{Z}G}_\chi$ has the Kaplansky property.*

0.4. The main results in the pro-p case

We consider the category of pro-p groups and show that a pro-p version of Rapaport's conjecture holds. Actually we show in Corollary 4 that a pro-p version of Theorem 1 holds. It is interesting to note that the proofs in the pro-p case do not involve heavy machinery as Σ-invariants and L^2-methods. By definition a pro-p module A in the category of pro-p $\mathbb{Z}_p[[G]]$-modules is of homological type FP_m if there is a projective resolution of A in this category with all modules in dimension at most m being (topologically) finitely generated. A pro-p group G is of homological type FP_m if the ring of the p-adic numbers $\mathbb{Z}_p$ considered as trivial pro-p $\mathbb{Z}_p[[G]]$-module is of homological type FP_m. When G is a pro-p group of cohomological dimension $cd_p(G) = n$ and homological type FP_n it has an Euler characteristics defined by

$$\chi_G = \sum_{i=0}^{cd(G)} (-1)^i dim_{\mathbb{F}_p}(\widehat{H}^i(G, \mathbb{F}_p)).$$

In order not to confuse the homology, cohomology, *Tor* and *Ext* functors in the pro-p category with the corresponding functors in the abstract category we use upper hat to denote the above functors in the category of pro-p groups and pro-p modules. Furthermore we use vcd_p and cd_p to denote the virtual cohomological dimension and the cohomological dimension of pro-p groups.

Theorem 4. *Let G be a pro-p group of cohomological dimension $cd_p(G) = n$, of homological type FP_n and let G have Euler characteristics* 0. *Suppose that N is a closed normal subgroup of G such that N is of homological type FP_{n-1}, $N \neq G$, $\mathbb{F}_p[[G/N]]$ is left and right Noetherian* (*abstractly or topologically is the same*) *and the completed group algebra $\mathbb{F}_p[[G/N]]$ does not have zero divisors.*

Then N is of homological type FP_n and G/N has finite virtual cohomological dimension

$$vcd_p(G/N) = cd_p(G) - cd_p(N).$$

In particular as G/N is infinite the virtual cohomological dimension of G/N is not zero, hence the cohomological dimension $cd_p(N)$ is at most $cd_p(G) - 1$.

Note that the pro-p completion of the example of a subgroup H of $G = F_2 \times F_2$ given after Theorem 1 shows that in Theorem 4 the condition on the Euler characteristics cannot be dropped in the case $cd_p(G) = 2$. As corollaries of Theorem 4 we obtain the following results.

Corollary 4. *Let G be a pro-p group of cohomological dimension $cd_p(G) = n$, of homological type FP_n and of Euler characteristics 0. Suppose that N is a normal closed subgroup of G such that N is of homological type FP_{n-1} and G/N is a pro-p group of finite rank. Then N is of homological type FP_n and G/N has finite virtual cohomological dimension*

$$vcd_p(G/N) = cd_p(G) - cd_p(N).$$

In particular either N has a finite index in G or N has cohomological dimension at most $cd_p(G) - 1$.

In [31, Corollary 3.3] it is shown that for a finitely presented pro-p group G of positive deficiency and of cohomological dimension $cd_p(G) = 2$ every finitely presented, normal, closed subgroup N of infinite index in G is a free pro-p group, G has deficiency 1 and either $N \simeq \mathbb{Z}_p$ or G/N is virtually procyclic. We go a bit further and show that a pro-p group of deficiency 1 that is (topologically finitely generated)-by-(infinite pro-p cyclic) has cohomological dimension at most 2.

Corollary 5. *Let G be a finitely presented pro-p group of deficiency 1 with a (topologically) finitely generated, normal, closed subgroup N such that $G/N \simeq \mathbb{Z}_p$. Then N is a free pro-p group. In particular the commutator subgroup G' is a free pro-p group and either G has cohomological dimension 2 or $G \simeq \mathbb{Z}_p$.*

1. Preliminaries on the homological invariants $\Sigma^m(G, A)$

Let G be a finitely generated group, $\chi : G \to \mathbb{R}$ be a non-zero character and A be a left $\mathbb{Z}[G]$-module of type FP_m. Then the homological invariant $\Sigma^m(G, A)$ as defined in [4] is

$$\Sigma^m(G, A) = \{[\chi] \in S(G) \mid A \text{ is of homological type } FP_m \text{ over } \mathbb{Z}[G_\chi]\},$$

where G_χ is the monoid $\{g \in G \mid \chi(g) \geqslant 0\}$ and $S(G)$ is the character sphere $\{[\chi] = \mathbb{R}_{>0}\chi \mid \chi \in Hom(G, \mathbb{R}) \setminus \{0\}\}$.

Theorem 5 (*[4, Theorem B]*). *Let G be a finitely generated group, H a subgroup of G that contains the commutator subgroup G' of G and A be a left $\mathbb{Z}[G]$-module of type FP_m. Then A is of type FP_m over $\mathbb{Z}[H]$ if and only if $S(G, H) = \{[\chi] \in S(G) \mid \chi(H) = 0\} \subseteq \Sigma^m(G, A)$.*

Let $\chi : G \to \mathbb{R}$ be a non-zero character. Then the Novikov completion $\widehat{\mathbb{Z}G}_\chi$ of $\mathbb{Z}[G]$ with respect to the character χ is a subset of $\mathbb{Z}^G = \prod_{g \in G} \mathbb{Z}$ and we write an element of $\prod_{g\in G} \mathbb{Z}$ as a formal, possibly infinite sum $\sum_{g \in G, z_g \in \mathbb{Z}} z_g g$. By definition

$$\widehat{\mathbb{Z}G}_\chi = \left\{ \sum_{g \in G, z_g \in \mathbb{Z}} z_g g \mid \text{ for any } j \text{ the set } \{g \in G \mid z_g \neq 0, \chi(g) \leqslant j\} \text{ is finite} \right\}.$$

Note that $\widehat{\mathbb{Z}G}_\chi$ is a ring with multiplication induced by the group operation in G and $\mathbb{Z}[G]$ is a subring of $\widehat{\mathbb{Z}G}_\chi$.

Theorem 6 (*[5, Theorem B.4.6]*). *Let A be a left $\mathbb{Z}[G]$-module of type FP_m and $\chi : G \to \mathbb{R}$ be a non-zero character. Then $[\chi] \in \Sigma^m(G, A)$ if and only if $Tor_i^{\mathbb{Z}[G]}(\widehat{\mathbb{Z}G}_\chi, A) = 0$ for all $i = 0, 1, \dots, m$.*

Actually in [5, Theorem B.4.6] a more general result is stated i.e. each of the conditions $[\chi] \in \Sigma^m(G, A)$ and $Tor_i^{\mathbb{Z}[G]}(\widehat{\mathbb{Z}G}_\chi, A) = 0$ for all $i = 0, 1, \dots, m$ is equivalent to a third one that we will not discuss here. A proof of a homotopy version of Theorem 6 is included in a more recent version of the book shown to me recently by Bieri and the completed proofs of [5, Theorem B.4.6] can be found in the diploma thesis [27]. Furthermore the Bieri–Strebel criterion in the case of discrete characters has a generalisation which treats non-acyclic complexes following Ranicki's approach [22] and was further generalised in [13].

2. Proofs of the results about the Kaplansky property

We start with a simple result.

Lemma 1. *Let G be a finitely generated finite-by-abelian group, $\chi : G \to \mathbb{R}$ a non-zero character. Then*

(a) *the Novikov ring $\widehat{\mathbb{Z}G}_\chi$ embeds in a matrix algebra over a commutative ring*;
(b) *the Novikov ring $\widehat{\mathbb{Z}G}_\chi$ has the Kaplansky property.*

Proof. (a) Note that as G is finite-by-abelian, it is central-by-finite. Let A be the centre of G and T be right transversal of A in G, so T is finite. A typical element of the Novikov ring $\widehat{\mathbb{Z}G}_\chi$ is a sum (in general infinite) $\sum_{a \in A, t \in T, z_{a,t} \in \mathbb{Z}} z_{a,t} at$ with support $\{at \mid z_{a,t} \neq 0\}$ having only finitely many elements of $\{g \in G \mid \chi(g) \leqslant j\}$ for every j. Rewriting $\sum_{a \in A, t \in T} z_{a,t} at$ as $\sum_{t \in T} (\sum_{a \in A} z_{a,t} a) t$ we get a decomposition

$$\widehat{\mathbb{Z}G}_\chi \simeq \bigoplus_{t \in T} \widehat{\mathbb{Z}A}_\chi t =: V.$$

Then $\widehat{\mathbb{Z}G}_\chi$ acts on the free left $\widehat{\mathbb{Z}A}_\chi$-module V (of rank $|T| = s$) via left multiplication in $\widehat{\mathbb{Z}G}_\chi$. This gives an embedding of the Novikov ring $\widehat{\mathbb{Z}G}_\chi$ into the matrix algebra $M_s(\widehat{\mathbb{Z}A}_\chi)$.

(b) Suppose B, C are elements of the matrix algebra $M_n(\widehat{\mathbb{Z}G_\chi})$. By part (a) $\widehat{\mathbb{Z}G_\chi}$ embeds in $M_s(\widehat{\mathbb{Z}A_\chi})$, hence $M_n(\widehat{\mathbb{Z}G_\chi})$ embeds in $M_{ns}(\widehat{\mathbb{Z}A_\chi})$. Finally we note that Kaplansky property always holds for matrices over commutative rings as there determinants and inverse matrices can be used i.e. thinking of B, C as elements of $M_{ns}(\widehat{\mathbb{Z}A_\chi})$ we have $BC = I_{ns}$ implies that $det(B)$ is an invertible element of $\widehat{\mathbb{Z}A_\chi}$, hence there is an inverse of B in $M_{ns}(\widehat{\mathbb{Z}A_\chi})$ that should be the right inverse C, so $CB = I_{ns}$. $\square$

Proof of Theorem 3. Let G be a finitely generated group, $\chi : G \to \mathbb{R}$ a non-zero character and N a finitely generated, residually finite subgroup of $Ker(\chi)$ such that $G' \subseteq N$. Let N_0 be a normal subgroup of N of finite index, say c. As N is finitely generated there are only finitely many normal subgroups of N of index c, so the intersection $N_1 = \bigcap_{g\in G} N_0^g$ is actually an intersection of finitely many subgroups of N of index c, hence is of finite index in N. In particular the intersection of all normal subgroups of G that are contained in N and have a finite index in N is trivial.

We claim that the Novikov ring $\widehat{\mathbb{Z}G_\chi}$ embeds in the inverse limit M of the Novikov rings $\widehat{\mathbb{Z}(G/H)}_{\chi_H}$, where the limit is over the normal subgroups H of G having finite index in N, with maps induced by the projection maps $G/H_1 \to G/H_2$ for $H_1 \subseteq H_2$ and $\chi_H : G/H \to \mathbb{R}$ being the character induced by χ. The projection maps $\widehat{\mathbb{Z}G_\chi} \to \widehat{\mathbb{Z}(G/H)}_{\chi_H}$ induce a homomorphism of rings $\theta : \widehat{\mathbb{Z}G_\chi} \to M$.

Let $\lambda = \sum_{g\in G, z_g \in \mathbb{Z}} z_g g$ be a non-zero element of $\widehat{\mathbb{Z}G_\chi}$. Consider $min\{\chi(g) \mid z_g \neq 0\} = s_0$ and let T_0 be the finite non-empty set $\{g \in G \mid z_g \neq 0, \chi(g) = s_0\} \subseteq$ the support of λ in G. By the first paragraph of the proof there is a subgroup N_1 of finite index in N such that N_1 is normal in G and the images of the elements of T_0 in G/N_1 are all different. Then the image of λ in $\widehat{\mathbb{Z}(G/N_1)}_{\chi_{N_1}}$ is not zero, hence θ is injective.

Finally we observe that by Lemma 1 the Kaplansky property holds for the rings $\widehat{\mathbb{Z}(G/H)}_{\chi_H}$ of the inverse system, hence the Kaplansky property holds for the inverse limit M.

3. Proof of Theorem 1 and its corollaries

3.1. Proof of Theorem 1

Let G be a group of geometric dimension n with a finite $K(G,1)$-complex Y of dimension n. Furthermore we suppose that Y has Euler characteristics zero. Then G acts freely on the universal covering space $\widetilde{Y}$ of Y and $\widetilde{Y}$ is a contractible CW-complex of dimension n. The cellular chain complex $\mathscr{F}$ of $\widetilde{Y}$ is a free resolution of the trivial $\mathbb{Z}[G]$-modules $\mathbb{Z}$

$$\begin{aligned}\mathscr{F} : 0 \to F_n \simeq \mathbb{Z}[G]^{m_n} &\xrightarrow{\partial_n} F_{n-1} \simeq \mathbb{Z}[G]^{m_{n-1}} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} F_0 \\ &\simeq \mathbb{Z}[G]^{m_0} \xrightarrow{\partial_0} \mathbb{Z} \longrightarrow 0,\end{aligned}$$

where m_i is the rank of the free $\mathbb{Z}[G]$-module F_i. In particular the cohomological dimension $cd(G)$ is at most n.

Let $\chi : G \to \mathbb{R}$ be a non-zero character such that $\chi(N) = 0$. Consider the Novikov's completion $\widehat{\mathbb{Z}G}_\chi$ of $\mathbb{Z}[G]$ at χ and the complex

$$\mathscr{C} = \widehat{\mathbb{Z}G}_\chi \otimes_{\mathbb{Z}[G]} \mathscr{F} : 0 \to C_n \simeq \widehat{\mathbb{Z}G}_\chi^{m_n} \xrightarrow{d_n} C_{n-1} \simeq \widehat{\mathbb{Z}G}_\chi^{m_{n-1}} \xrightarrow{d_{n-1}} \cdots$$
$$\xrightarrow{d_1} C_0 \simeq \widehat{\mathbb{Z}G}_\chi^{m_0} \xrightarrow{d_0} \widehat{\mathbb{Z}G}_\chi \otimes_{\mathbb{Z}[G]} \mathbb{Z} \simeq 0 \longrightarrow 0,$$

where $d_i = id_{\widehat{\mathbb{Z}G}_\chi} \otimes_{\mathbb{Z}[G]} \partial_i$. As N is of homological type FP_{n-1} by Theorem 5 $[\chi] \in \Sigma^{n-1}(G, \mathbb{Z})$. Then by Theorem 6

$$Tor_j^{\mathbb{Z}[G]}(\widehat{\mathbb{Z}G}_\chi, \mathbb{Z}) = 0 \quad \text{for all } j = 0, 1, \ldots, n-1,$$

so the complex $\mathscr{C}$ is exact in dimensions $-1, 0, 1, \ldots, n-1$. Our aim is to show that d_n is injective. This will imply that $Tor_n^{\mathbb{Z}[G]}(\widehat{\mathbb{Z}G}_\chi, \mathbb{Z}) = 0$ and so by Theorem 6 $[\chi] \in \Sigma^n(G, \mathbb{Z})$. As this is true for arbitrary $[\chi] \in S(G, N)$ by Theorem 5 N is of homological type FP_n, as required.

Now we prove that the map d_n is injective. First we define P_i to be the image of d_{i+1} for $0 \leqslant i \leqslant n-1$. Note that $P_0 = C_0$ is a free $\widehat{\mathbb{Z}G}_\chi$-module, hence $C_1 \simeq P_0 \oplus P_1$ and P_1 is a projective $\widehat{\mathbb{Z}G}_\chi$-module. Using induction on i together with the fact that the complex $\mathscr{C}$ is exact in all dimensions $-1, 0, 1, \ldots, n-1$ we deduce that

$$C_i \simeq P_i \oplus P_{i-1} \quad \text{for } i = 1, 2, \ldots, n-1.$$

Now define the surjective homomorphism of $\widehat{\mathbb{Z}G}_\chi$-modules

$$\begin{aligned}\theta : B = C_n \oplus P_{n-2} \oplus P_{n-3} \oplus \cdots \oplus P_1 \oplus P_0 &\to D \\ &= P_{n-1} \oplus P_{n-2} \oplus \cdots \oplus P_1 \oplus P_0\end{aligned}$$

by $\theta = d_n \oplus id_{P_{n-2}} \oplus \cdots \oplus id_{P_1} \oplus id_{P_0}$ i.e. all the maps in the direct sum are the identity ones except for the first one that is d_n. Note that if n is even

$$\begin{aligned}D &\simeq (P_{n-1} \oplus P_{n-2}) \oplus \cdots \oplus (P_1 \oplus P_0) \simeq C_{n-1} \oplus \cdots \oplus C_3 \oplus C_1 \\ &\simeq \widehat{\mathbb{Z}G}_\chi^{m_{n-1}+\cdots+m_3+m_1}\end{aligned}$$

and

$$\begin{aligned}B &= C_n \oplus (P_{n-2} \oplus P_{n-3}) \oplus \cdots \oplus (P_2 \oplus P_1) \oplus P_0 \\ &\simeq C_n \oplus C_{n-2} \oplus \cdots \oplus C_2 \oplus C_0 \\ &\simeq \widehat{\mathbb{Z}G}_\chi^{m_n+m_{n-2}+\cdots+m_2+m_0}.\end{aligned}$$

If n is odd

$$\begin{aligned}D &\simeq (P_{n-1} \oplus P_{n-2}) \oplus \cdots \oplus (P_2 \oplus P_1) \oplus P_0 \\ &\simeq C_{n-1} \oplus C_{n-3} \oplus \cdots \oplus C_2 \oplus C_0 \\ &\simeq \widehat{\mathbb{Z}G}_\chi^{m_{n-1}+m_{n-3}+\cdots+m_2+m_0}\end{aligned}$$

and

$$\begin{aligned} B &= C_n \oplus (P_{n-2} \oplus P_{n-3}) \oplus \cdots \oplus (P_1 \oplus P_0) \\ &\simeq C_n \oplus C_{n-2} \oplus \cdots \oplus C_3 \oplus C_1 \\ &\simeq \widehat{\mathbb{Z}G}_\chi^{m_n+m_{n-2}+\cdots+m_3+m_1}. \end{aligned}$$

Note that as the Euler characteristics of Y is 0

$$\sum_{i\ par} m_i = \sum_{i\ impar} m_i = \alpha,$$

thus in both cases

$$B \simeq \widehat{\mathbb{Z}G}_\chi^\alpha \simeq D.$$

Finally as θ is a homomorphism of free $\widehat{\mathbb{Z}G}_\chi$-modules of the same rank α it can be represented by a matrix $M_1 \in M_\alpha(\widehat{\mathbb{Z}G}_\chi)$ i.e. the coordinates of $\theta(b)$ for some $b \in B$ are the elements of the matrix-line $\bar{b}M_1$, where $\bar{b}$ is the matrix-line with entries the coordinates of b. The fact that θ is surjective is equivalent with the existence of a matrix $M_2 \in M_\alpha(\widehat{\mathbb{Z}G}_\chi)$ such that M_2M_1 is the identity matrix I_α. By Theorem 3 $\widehat{\mathbb{Z}G}_\chi$ satisfies the Kaplansky property, hence $M_1M_2 = I_\alpha$, in particular θ is injective. Then d_n is injective as required.

3.2. Proof of Corollary 1

The following Proposition 1 stems from the work of Bieri in [3,2]. The case $n = 2$ is [2, Theorem B + Remark 5.4] and the case $n > 2$ under some more restrictive hypothesis for N but without the hypothesis that G/N is abelian is done in [3, Theorem 8.4]. Corollary 1 follows from Theorem 1 and Proposition 1.

Proposition 1. *Let G be a finitely generated group of cohomological dimension $cd(G) = n$ and N be a normal subgroup of G of homological type FP_n such that G/N is abelian. Then G/N has finite virtual cohomological dimension $vcd(G/N) = cd(G) - cd(N)$.*

Proof. Note that G/N is a finitely generated abelian group, so direct sum of a finite abelian group with a free abelian group. As the property FP_m and the cohomological dimension are preserved when going down to a subgroup of finite index we can substitute G with any subgroup G_1 such that $N \subseteq G_1$ and G/G_1 is finite. So without loss of generality we can assume that G/N is free abelian, say

$$G/N \simeq \mathbb{Z}^m \quad \text{for some } m \geqslant 0.$$

If $m = 0$ there is nothing to prove, so we can assume that $m \geqslant 1$.

Let $cd(N) = k$, as $cd(N) \leqslant cd(G) = n$ the group N is of homological type FP_k. By [3, Proposition 5.1] there is a free $\mathbb{Z}[N]$-module F such that $H^k(N, F) \neq 0$. Furthermore by [3, Theorem 1.3] as N is of type FP_k the functor $H^k(N, \_)$ commutes with direct limits, in particular commutes with direct sums and so $H^k(N, F)$ is a direct sum of copies of $H^k(N, \mathbb{Z}[N])$, so

$$H^k(N, \mathbb{Z}[N]) \neq 0.$$

As in [3, p. 117] the fact that N is of type FP_k implies the existence of an isomorphism of (left) $\mathbb{Z}[G/N]$-modules

$$H^k(N, \mathbb{Z}[G]) \simeq \mathbb{Z}[G/N] \otimes_{\mathbb{Z}} H^k(N, \mathbb{Z}[N]),$$

where the $\mathbb{Z}[G/N]$ action on the tensor product is via left multiplication on the component $\mathbb{Z}[G/N]$, the $\mathbb{Z}[G/N]$ action on the left-hand side of the isomorphism is the one used in the calculation of the $E_2^{*,k}$-term of the Lyndon–Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G/N, H^q(N, \mathbb{Z}[G])) \underset{p}{\Longrightarrow} H^{p+q}(G, \mathbb{Z}[G]).$$

We observe that the original argument in [3, p. 117] is for right modules but the modification from right to left modules is obvious. It is well known that

$$cd(G) = n \leqslant cd(G/N) + cd(N) = m + k.$$

Here we show that the equality holds when N is of type FP_k. As $G/N \simeq \mathbb{Z}^m$ is a Poincare duality group of dimension m we have $H^j(G/N, \mathbb{Z}[G/N]) = 0$ for $j \neq m$ and $H^m(G/N, \mathbb{Z}[G/N]) \simeq \mathbb{Z}$. Note that for $0 \leqslant p \leqslant m$, $0 \leqslant q \leqslant k$

$$\begin{aligned} E_2^{p,q} &= H^p(G/N, H^q(N, \mathbb{Z}[G])) = H^p(G/N, \mathbb{Z}[G/N] \otimes_{\mathbb{Z}} H^q(N, \mathbb{Z}[N])) \\ &\simeq H^p(G/N, \mathbb{Z}[G/N]) \otimes_{\mathbb{Z}} H^q(N, \mathbb{Z}[N]) \end{aligned}$$

and for $p > m$ or $q > k$

$$E_2^{p,q} = 0.$$

So the spectral sequence $\{E_2^{p,q}\}_{p,q}$ has only one non-zero column $\{E_2^{m,q}\}_q$ and hence collapses i.e.

$$E_2^{m,q} \simeq E_\infty^{m,q} \simeq H^{m+q}(G, \mathbb{Z}[G]) \quad \text{for any } q \geqslant 0.$$

Then

$$\begin{aligned} H^{m+k}(G, \mathbb{Z}[G]) \simeq E_2^{m,k} &= H^m(G/N, \mathbb{Z}[G/N]) \otimes_{\mathbb{Z}} H^k(N, \mathbb{Z}[N]) \\ &\simeq \mathbb{Z} \otimes_{\mathbb{Z}} H^k(N, \mathbb{Z}[N]) \simeq H^k(N, \mathbb{Z}[N]) \neq 0, \end{aligned}$$

so $cd(G) \geqslant m + k$. □

3.3. *Proof of Corollary 2*

By [12, Theorem 2 + Lemma, p. 273] G is of geometric dimension at most 2, so of cohomological dimension at most 2. If G is not of cohomological dimension 2 then it should be of cohomological dimension 1, so is free by the Swan-Stalling's result [8, Theorem 3.15, p. 114]. Furthermore as G has deficiency one $G \simeq \mathbb{Z}$ and the commutator subgroup G' is trivial.

If G has cohomological dimension 2 then it has geometric dimension 2 and by Corollary 1 for the finitely generated group $N = G'$ the commutator subgroup G' has cohomological dimension one, hence is free.

3.4. Proof of Theorem 2 and Corollary 3

Let $\widetilde{X}$ be the universal covering of X. Then the cellular chain complex of $\widetilde{X}$ over $\mathbb{Z}$ is

$$\mathscr{F}: 0 \to F_n \simeq \mathbb{Z}[G]^{m_n} \xrightarrow{\partial_n} F_{n-1} \simeq \mathbb{Z}[G]^{m_{n-1}} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} F_0 \simeq \mathbb{Z}[G]^{m_0} \xrightarrow{\partial_0} \mathbb{Z} \longrightarrow 0,$$

where m_i is the number of cells of X of dimension i. Let $[\chi] \in \Sigma^\infty(G, \mathbb{Z})$. Then by Theorem 6 $Tor_i^{\mathbb{Z}[G]}(\widehat{\mathbb{Z}G}_\chi, \mathbb{Z}) = 0$ for all i. In particular the complex

$$\widehat{\mathbb{Z}G}_\chi \otimes_{\mathbb{Z}[G]} \mathscr{F} : 0 \to \widehat{\mathbb{Z}G}_\chi^{m_n} \xrightarrow{d_n} \widehat{\mathbb{Z}G}_\chi^{m_{n-1}} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} \widehat{\mathbb{Z}G}_\chi^{m_0} \xrightarrow{d_0} \widehat{\mathbb{Z}G}_\chi \otimes_{\mathbb{Z}[G]} \mathbb{Z} \simeq 0$$

is exact, where $d_i = id_{\widehat{\mathbb{Z}G}_\chi} \otimes_{\mathbb{Z}[G]} \partial_i$. Finally we note that the rank of a finitely generated free $\widehat{\mathbb{Z}G}_\chi$-module is uniquely determined as $\widehat{\mathbb{Z}G}_\chi$ has a quotient that is an integral domain i.e. let Q be the quotient of the abelianization $G_{ab} = G/[G, G]$ modulo the torsion part of G_{ab}, then χ induces a character χ_0 of Q and $\widehat{\mathbb{Z}Q}_{\chi_0}$ is a domain and a quotient of $\widehat{\mathbb{Z}G}_\chi$. Then we can apply [25, Lemma 2.7] to deduce $\sum_i (-1)^i m_i = 0$.

Finally Corollary 3 follows from Theorems 2 and 5. Theorem 5 guarantees that every $[\chi] \in S(G, N)$ is an element of $\Sigma^\infty(G, \mathbb{Z})$, hence $\Sigma^\infty(G, \mathbb{Z}) \neq \emptyset$.

4. Preliminaries on pro-p groups

In this section we consider some facts about pro-p groups that will be used in the proof of Theorem 4. Good references for the categories of pro-p $\mathbb{Z}_p[[G]]$-modules and discrete $\mathbb{Z}_p[[G]]$-modules with continuous G-action are [24,28,32], here $\mathbb{Z}_p[[G]]$ denotes the completed group algebra of G with coefficient ring $\mathbb{Z}_p$ i.e.

$$\mathbb{Z}_p[[G]] = \varprojlim_{i, U} (\mathbb{Z}/p^i\mathbb{Z})[G/U],$$

where the inverse limit is over all positive integers i and all open normal subgroups U of G. We note that the modules if not otherwise stated are left ones and that in the categories mentioned above all G-actions are continuous. In order not to confuse the homology, cohomology, *Hom*, *Tor* and *Ext* functors in the pro-p category with the corresponding functors in the abstract category we use upper hat to denote the above functors in the category of pro-p groups and pro-p modules. Still if we work with a complex $\mathscr{C}$ of pro-p modules we will use the traditional notation $H_i(\mathscr{C})$ or $H^i(\mathscr{C})$ for the homology or cohomology groups of the complex. Note that in this case the traditional homology $H_i(\mathscr{C})$ has an induced pro-p structure as it is a quotient of a kernel by an image of some continuous maps of compacts.

We use the notation $\widehat{\otimes}$ for the completed tensor product in the category of pro-p modules and the traditional notation $\otimes$ for the abstract tensor product. By [24, Proposition 5.5.3d] for a profinite ring R and profinite R-modules V_1 and V_2 if one of the modules is (topologically) finitely generated there is a natural isomorphism $V_1 \widehat{\otimes}_R V_2 \simeq V_1 \otimes_R V_2$.

By definition the cohomological dimension $cd_p(G)$ of a pro-p group G is

$$supp\{k \mid \text{there is a discrete } \mathbb{Z}_p[[G]]\text{-module } W \text{ with } \widehat{H}^k(G, W) \neq 0\}.$$

Here $\widehat{H}^k(G, \ \ ) = \widehat{Ext}^k_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, \ \ )$ is the cohomology functor, the kth (right) derived functor of $\widehat{Hom}_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, \ \ )$, that can be calculated using either projective resolution of the trivial (left) $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p$ in the category of (left) pro-p $\mathbb{Z}_p[[G]]$-modules or injective resolution of W in the category of (left) discrete $\mathbb{Z}_p[[G]]$-modules with continuous G-action to get $\widehat{H}^k(G, W)$. The categories of pro-p $\mathbb{Z}_p[[G]]$-modules and discrete $\mathbb{Z}_p[[G]]$-modules with continuous G-action are dual via the Pontryagin duality [24, Section 5.1]. The homology group $\widehat{H}_k(G, \ \ ) = \widehat{Tor}_k^{\mathbb{Z}_p[[G]]}(\ \ , \mathbb{Z}_p)$ is the kth (left) derived functor of the completed tensor product functor $\_\_\widehat{\otimes}_{\mathbb{Z}_p[[G]]}\mathbb{Z}_p$ and can be calculated via projective resolutions of the trivial (left) pro-p $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p$ or via a projective resolution of V in the category of (right) pro-p $\mathbb{Z}_p[[G]]$-modules to obtain $\widehat{H}_k(G, V)$.

We note that by [32, Proposition 7.5.1] if R is a profinite local ring every finitely generated (abstractly or topologically is the same) profinite projective R-modules is free. In particular if G is a pro-p group and k is a commutative pro-p ring the completed group algebra $k[[G]]$ is a local ring and every finitely generated projective $k[[G]]$-module is free [32, Corollary 7.5.4]. By definition a topological module is topologically Noetherian if every non-empty family of closed submodules has a maximal element. In the case of a profinite local ring R a profinite R-module M is topologically Noetherian if and only if it is abstractly Noetherian [32, Proposition 8.6.4]. A pro-p group G is said to be of homological type FP_m if the trivial pro-p $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p$ has a projective resolution of pro-p $\mathbb{Z}_p[[G]]$-modules such that all modules in dimension $\leqslant m$ are finitely generated (topologically or abstractly is the same). In the following lemmas we collect several well-known results.

Lemma 2. *Let G be a pro-p group and N be a closed normal subgroup of G such that $\mathbb{Z}_p[[G/N]]$ is left and right Noetherian* (*abstractly or topologically is the same*). *Then the following are equivalent*:

(a) *G is of homological type FP_m*;
(b) *the homology groups $\widehat{H}_i(N, \mathbb{Z}_p)$ are* (*topologically*) *finitely generated as pro-p $\mathbb{Z}_p[[G/N]]$-modules for $i = 1, \dots, m$, where the G/N action on the homology groups is induced by the action of G on N via conjugation. In particular for $N = G$ we have $\widehat{H}_i(G, \mathbb{Z}_p)$ is* (*topologically*) *finitely generated as an abelian pro-p group for $i = 1, \dots, m$*;
(c) *the homology groups $\widehat{H}_i(N, \mathbb{F}_p)$ are* (*topologically*) *finitely generated as pro-p $\mathbb{F}_p[[G/N]]$-modules for $i = 1, \dots, m$, where the G/N action on the homology groups is induced by the action of G on N via conjugation. In particular for $N = G$ we have $\widehat{H}_i(G, \mathbb{F}_p)$ is finite for $i = 1, \dots, m$*;

(d) *the cohomology groups* $\widehat{H}^i(G, V)$ *are finite for every finite discrete* $\mathbb{Z}_p[[G]]$*-module* V *with continuous* G*-action and for every* $i = 1, \ldots, m$;
(e) *the cohomology groups* $\widehat{H}^i(G, \mathbb{F}_p)$ *are finite for* $i = 1, \ldots, m$.

Remark. The equivalence between (a) and (c) requires only that $\mathbb{F}_p[[G/N]]$ is left and right Noetherian.

Proof. The equivalence between (a)–(c) follows from [15, Section 3, Theorem 2]. The equivalence between (a) and (d) is shown in [30, Proposition 4.2.3]. Finally we show that (e) implies (d). Indeed by [24, Lemma 7.1.5] every simple discrete $\mathbb{Z}_p[[G]]$-module with continuous G-action is isomorphic to $\mathbb{F}_p$ and so every finite discrete $\mathbb{Z}_p[[G]]$-module V with continuous G-action has a finite filtration of discrete $\mathbb{Z}_p[[G]]$-submodules with factors isomorphic to $\mathbb{F}_p$. Then by induction on the length of this filtration that is in fact the dimension of V as a vector space over $\mathbb{F}_p$ there is a $\mathbb{Z}_p[[G]]$-submodule W of V such that V/W is isomorphic to the trivial $\mathbb{Z}_p[[G]]$-module $\mathbb{F}_p$ and all cohomology groups $\widehat{H}^i(G, W)$ are finite for $i = 1, \ldots, m$. Then by the long exact sequence in cohomology

$$0 \to \widehat{H}^0(G, W) \to \widehat{H}^0(G, V) \to \widehat{H}^0(G, \mathbb{F}_p) \to \widehat{H}^1(G, W) \to \cdots$$
$$\to \widehat{H}^i(G, W) \to \widehat{H}^i(G, V) \to \widehat{H}^i(G, \mathbb{F}_p) \to \widehat{H}^{i+1}(G, W) \to \cdots$$

together with the fact that for every $i = 1, \ldots, m$ and for $\widetilde{V} = W$ and $\widetilde{V} = \mathbb{F}_p$ the cohomology groups $\widehat{H}^i(G, \widetilde{V})$ are finite, we deduce that (d) holds. □

Lemma 3. *Let* G *be a pro-*p *group with an open subgroup* H. *Then*

(a) *if* G *is of homological type* FP_m *the group* H *is of homological type* FP_m;
(b) *if* G *is of finite cohomological dimension* $cd_p(G) = n$ *the group* H *is of finite cohomological dimension* $cd_p(H) = n$;
(c) *if* G *is a group of Euler characteristics* 0 *the group* H *has Euler characteristics* 0.

Proof. (a) This is [30, Proposition 4.2.1].

(b) This is a partial case of [24, Theorem 7.3.1].

(c) It follows from the equality $\chi_H = [G : H]\chi_G$, where χ_H and χ_G are the Euler characteristics of H and G respectively [28, I.4.1, Example (b)]. □

We finish the preliminaries on pro-p groups revising Lazard's classification of p-adic analytic pro-p groups [16] and the uniformly powerful group approach [17] as stated in [9]. We remark that a powerful finitely generated pro-p group is uniform if and only if it is torsion-free [9, Theorem 4.8]. We say that a pro-p group is of finite rank if it satisfies the condition of [9, Chapter 3.2].

Theorem 7 (*Dixon et al.* [9, Corollary 9.3.6]). *For a topological group* G *the following are equivalent*:

(a) G *is a compact* p*-adic analytic group*;
(b) G *contains a normal*, *open*, *uniformly powerful pro-*p *subgroup of finite index*;
(c) G *is a finitely generated profinite group containing an open subgroup which is a pro-*p *group of finite rank.*

5. Proof of Theorem 4 and its corollaries

5.1. Proof of Theorem 4

Let G and N be pro-p groups satisfying the assumptions of Theorem 4.

Lemma 4. *There is a free resolution of the trivial left $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p$ with finitely generated free pro-p $\mathbb{Z}_p[[G]]$-modules*

$$\cdots \xrightarrow{\partial_{i+1}} F_i \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \longrightarrow \mathbb{Z}_p \longrightarrow 0$$

such that $F_j = 0$ for all $j \geqslant n+1$ and every F_i is of finite rank m_i such that the alternating sum $m_0 - m_1 + m_2 - \cdots$ is zero.

Proof. As G is of homological type FP_n there is a projective resolution of $\mathbb{Z}_p$ in the category of pro-p $\mathbb{Z}_p[[G]]$-modules

$$\cdots \xrightarrow{\partial_{i+1}} F_i \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} \mathbb{Z}_p \longrightarrow 0$$

with all F_i (topologically) finitely generated for $i \leqslant n$. Note that the short exact sequence

$$0 \to Im\,\partial_{i+1} \simeq Ker\,\partial_i \to F_i \to Im\,\partial_i \to 0$$

gives a long exact sequence in $\widehat{Ext}$ and so for every discrete $\mathbb{Z}_p[[G]]$-module V with continuous G-action and $j \geqslant 1$

$$\widehat{Ext}^j_{\mathbb{Z}_p[[G]]}(Im\,\partial_{i+1}, V) \simeq \widehat{Ext}^{j+1}_{\mathbb{Z}_p[[G]]}(Im\,\partial_i, V).$$

Then for $P = Im\,\partial_n = Ker\,\partial_{n-1}$ and $j \geqslant 1$ we have

$$\widehat{Ext}^j_{\mathbb{Z}_p[[G]]}(P, V) \simeq \widehat{Ext}^{j+n}_{\mathbb{Z}_p[[G]]}(Im\,\partial_0 = \mathbb{Z}_p, V) = 0,$$

where the last inequality comes from the fact that the cohomological dimension $cd_p(G) = n$. Hence the functor $\widehat{Hom}_{\mathbb{Z}_p[[G]]}(P, \_)$ in the category of discrete $\mathbb{Z}_p[[G]]$-modules with continuous G-action is exact and P is a projective pro-p $\mathbb{Z}_p[[G]]$-module. Thus there is an exact complex

$$\mathscr{F} : 0 \longrightarrow P \longrightarrow F_{n-1} \xrightarrow{\partial_{n-1}} F_{n-2} \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} \mathbb{Z}_p \longrightarrow 0.$$

By an obvious pro-p version of [6, Chapter 8, Proposition 4.3] stated in [30, Proposition 3.7.1] since G is of type FP_n and all $F_0, F_1, \ldots, F_{n-1}$ are (topologically) finitely generated projective pro-p $\mathbb{Z}_p[[G]]$-modules, P is finitely generated as pro-p $\mathbb{Z}_p[[G]]$-module. Finally we note that as $\mathbb{Z}_p[[G]]$ is a local ring every finitely generated (topologically or abstractly is the same) pro-p $\mathbb{Z}_p[[G]]$-module is a free pro-p $\mathbb{Z}_p[[G]]$-module [32, Corollary 7.5.4], in particular F_i is free of finite rank m_i for $i \leqslant n$ where by abuse of notation $P = F_n$.

Let $\widetilde{\mathscr{F}}$ be the deleted resolution obtained from $\mathscr{F}$ by deleting $\mathbb{Z}_p$. Consider the complex $\mathbb{F}_p \otimes_{\mathbb{Z}_p[[G]]} \widetilde{\mathscr{F}}$, then its ith dimensional module is $\mathbb{F}_p^{m_i}$ for $0 \leqslant i \leqslant n$ and the alternating sum of the dimensions of the homologies of $\mathbb{F}_p \otimes_{\mathbb{Z}_p[[G]]} \widetilde{\mathscr{F}}$ is zero since the Euler characteristics of G is zero. Then the alternating sum $m_0 - m_1 + m_2 - \cdots$ is zero as required. $\square$

Let N be the normal closed subgroup of G given by the assumptions of Theorem 4 and define $R = \mathbb{F}_p[[G/N]]$. We define an abstract ring S as being the skew ring of fractions of R. This is possible as $\mathbb{F}_p[[G/N]]$ is without zero divisors and is left and right Noetherian [10, Chapter 9].

Proposition 2. *The homology group* $\widehat{H}_n(N, \mathbb{F}_p) = \widehat{Tor}_n^{\mathbb{Z}_p[[N]]}(\mathbb{F}_p, \mathbb{Z}_p) = 0$. *In particular* N *is of homological type* FP_n.

Proof. Consider a free resolution of the trivial left pro-p $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p$ given by Lemma 4

$$0 \longrightarrow F_n \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_{n-1}} F_{n-2} \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_1} F_0 \longrightarrow \mathbb{Z}_p \longrightarrow 0$$

i.e. F_i is free pro-p $\mathbb{Z}_p[[G]]$-module of rank $m_i \geqslant 0$ and the alternating sum $m_0 - m_1 + m_2 - \cdots$ is zero. We apply $\mathbb{F}_p \widehat{\otimes}_{\mathbb{Z}_p[[N]]}$ to this complex and obtain a complex

$$\mathscr{K} : 0 \longrightarrow R^{m_n} \xrightarrow{d_n} \cdots \xrightarrow{d_2} R^{m_1} \xrightarrow{d_1} R^{m_0} \xrightarrow{d_0} \mathbb{F}_p \longrightarrow 0,$$

where $R = \mathbb{F}_p[[G/N]]$. By Lemma 2 as N is of homological type FP_{n-1} the homology groups $H_i(\mathscr{K}) \simeq \widehat{H}_i(N, \mathbb{F}_p)$ are finite for all $i \leqslant n-1$. Note that if V is a (left) R-module and V is finite dimensional vector space over $\mathbb{F}_p$ then $S \otimes_R V = 0$. Indeed there exists an element of infinite order $\bar{g} \in (G/N)\backslash\{1\}$ such that $\bar{g}^z - 1$ is invertible in S for every $z \in Z\backslash\{0\}$ and as V is finite there is some $j > 0$ such that $\bar{g}^j - 1$ acts like zero on V.

Then applying the functor $S \otimes_R$ to $\mathscr{K}$ we obtain a complex $\mathscr{C} = S \otimes_R \mathscr{K}$ of abstract free (left) S-modules

$$0 \to C_n = S^{m_n} \xrightarrow{id_S \otimes_R d_n} C_{n-1} = S^{m_{n-1}} \to \cdots \xrightarrow{id_S \otimes_R d_1} C_0 = S^{m_0} \to 0 = S \otimes_R \mathbb{F}_p \to 0$$

with $H_i(\mathscr{C}) = 0$ for $i = 0, 1, \ldots, n-1$ since $H_i(\mathscr{K})$ is finite for every $i \leqslant n-1$. Since the alternating sum $m_0 - m_1 + m_2 - \cdots$ is zero the argument from the proof of Theorem 1 can be used to show that the map $id_S \otimes_R d_n$ from the above complex is injective i.e. define P_i as the image of $id_S \otimes_R d_{i+1}$ and define a surjective homomorphism of abstract S-modules

$$\begin{aligned} \theta : B &= C_n \oplus P_{n-2} \oplus P_{n-3} \oplus \cdots \oplus P_1 \oplus P_0 \to D \\ &= P_{n-1} \oplus P_{n-2} \oplus \cdots \oplus P_1 \oplus P_0 \end{aligned}$$

by $\theta = (id_{S_n} \otimes_R d_n) \oplus id_{P_{n-2}} \oplus \cdots \oplus id_{P_1} \oplus id_{P_0}$. Observe that $C_0 = P_0$, $C_i \simeq P_i \oplus P_{i-1}$ for $1 \leqslant i \leqslant n-1$ and $P_0, P_1, \ldots, P_{n-1}$ are abstract projective S-modules. Then $B \simeq D \simeq S^\alpha$, where $\alpha = \sum_{i\ even} m_i = \sum_{i\ odd} m_i$.

Finally we note that S is a division ring, hence its unique left ideals are 0 and R. In particular S is a left hereditary ring and by [26, Chapter 4, Theorem 4.17] every submodule of a free S-module is a direct sum of copies of left ideals of S, so is free. Thus the kernel of θ is a free S-module, say or rank k, so $S^\alpha \simeq B \simeq D \oplus Ker(\theta) \simeq S^{\alpha+k}$. By the Jordan–Holder theorem the rank of finitely generated free modules over division rings is uniquely determined and equals the number of factors in a composition series. In particular $k=0$ and θ and $id_S\otimes_R d_n$ are injective.

Then we have a commutative diagram

$$\begin{array}{ccc} S^{m_n} & \xrightarrow{id_S\otimes_R d_n} & S^{m_{n-1}} \\ \uparrow & & \uparrow \\ R^{m_n} & \xrightarrow{d_n} & R^{m_{n-1}} \end{array}$$

where the vertical maps are the injective maps induced by the inclusion of R in S. As $id_S\otimes_R d_n$ is injective the map d_n is injective, so $\widehat{H}_n(N,\mathbb{F}_p)=\widehat{Tor}_n^{\mathbb{Z}_p[[N]]}(\mathbb{F}_p,\mathbb{Z}_p)\simeq H_n(\mathscr{K})=0$. In particular as N is of homological type FP_{n-1} and $\widehat{H}_n(N,\mathbb{F}_p)=0$ is finite, by Lemma 2 N is of homological type FP_n, as required. $\square$

Finally we finish the proof of Theorem 4. By Proposition 2 N is of type FP_n. Then by Lemma 2 the cohomology group $\widehat{H}^i(N,\mathbb{F}_p)$ is finite for all $1\leqslant i\leqslant n$ and by [31, Theorem A] G/N has finite virtual cohomological dimension $vcd_p(G/N)=cd_p(G)-cd_p(N)$. In particular as G/N is infinite, $vcd_p(G/N)\geqslant 1$ and $cd_p(N)=cd_p(G)-vcd_p(G/N)\leqslant cd_p(G)-1$.

We finish this section with a direct proof of the inequality $cd_p(N)\leqslant n-1$ that does not involve the results from [31]. It is well known that a pro-p group H is of cohomological dimension $cd_p(H)\leqslant m$ if and only if $\widehat{H}^{m+1}(H,\mathbb{F}_p)=0$ [24, Corollary 7.1.6]. We give homological version of this result in Lemma 5. By Lemma 5 for $m=n-1$ and $H=N$ we get directly that $cd_p(N)\leqslant n-1$. We note that Lemma 5 will be needed in the proof of Corollary 5. We remind the reader that by our definition of homology $\widehat{H}_i(H,\_)=\widehat{Tor}_i^{\mathbb{Z}_p[[H]]}(\_,\mathbb{Z}_p)$.

Lemma 5. *Let H be a pro-p group such that*

$$\widehat{H}_{m+1}(H,\mathbb{F}_p)=\widehat{Tor}_{m+1}^{\mathbb{Z}_p[[H]]}(\mathbb{F}_p,\mathbb{Z}_p)=0.$$

Then H has finite cohomological dimension $cd_p(H)\leqslant m$.

Proof. Let

$$0\longrightarrow M\longrightarrow P_m\xrightarrow{\partial_m}\cdots\to P_1\xrightarrow{\partial_1}P_0\xrightarrow{\partial_0}\mathbb{Z}_p\to 0$$

be an exact complex of left pro-p $\mathbb{Z}_p[[H]]$-modules with $P_0,P_1,\ldots,P_m$ projective and $\mathbb{Z}_p$ being the trivial module. To prove that $cd_p(H)\leqslant m$ is sufficient to show that $M=0$, as the above complex will be a projective resolution and can be used to calculate $\widehat{H}^{m+1}(H,D)=0$ for every discrete $\mathbb{Z}_p[[H]]$-module D with continuous H-action. We will show that some restrictions on the map ∂_m enforce $M=0$.

We show by induction on $0\leqslant i\leqslant m$ that $\widehat{Tor}_{m+1-i}^{\mathbb{Z}_p[[H]]}(\mathbb{F}_p,Im(\partial_i))=0$, the case $i=0$ being given by the hypothesis of the lemma. Indeed there is a short exact sequence of pro-p $\mathbb{Z}_p[[H]]$-modules

$$0\longrightarrow Im(\partial_{i+1})=Ker(\partial_i)\longrightarrow P_i\xrightarrow{\partial_i}Im(\partial_i)\longrightarrow 0$$

that induces a long exact sequence in homology and for $0 \leqslant i < m$

$$\cdots \longrightarrow \widehat{Tor}_{m+1-i}^{\mathbb{Z}_p[[H]]}(\mathbb{F}_p, P_i) = 0 \longrightarrow \widehat{Tor}_{m+1-i}^{\mathbb{Z}_p[[H]]}(\mathbb{F}_p, Im(\partial_i)) \xrightarrow{\theta_i}$$
$$\widehat{Tor}_{m-i}^{\mathbb{Z}_p[[H]]}(\mathbb{F}_p, Im(\partial_{i+1})) \longrightarrow \widehat{Tor}_{m-i}^{\mathbb{Z}_p[[H]]}(\mathbb{F}_p, P_i) = 0 \longrightarrow \cdots$$

In particular the connecting homomorphism θ_i is an isomorphism of abelian pro-p groups for $0 \leqslant i < m$ and hence

$$\widehat{Tor}_1^{\mathbb{Z}_p[[H]]}(\mathbb{F}_p, Im(\partial_m)) \simeq \cdots \simeq \widehat{Tor}_{m+1}^{\mathbb{Z}_p[[H]]}(\mathbb{F}_p, Im(\partial_0)) = 0.$$

Finally consider the short exact sequence of pro-p $\mathbb{Z}_p[[H]]$-modules

$$0 \longrightarrow M \longrightarrow P_m \longrightarrow Im(\partial_m) = Ker(\partial_{m-1}) \longrightarrow 0$$

and the corresponding long exact sequence in homology

$$\cdots \longrightarrow 0 = \widehat{Tor}_1^{\mathbb{Z}_p[[H]]}(\mathbb{F}_p, Im(\partial_m)) \longrightarrow \mathbb{F}_p \widehat{\otimes}_{\mathbb{Z}_p[[H]]} M \longrightarrow$$
$$\mathbb{F}_p \widehat{\otimes}_{\mathbb{Z}_p[[H]]} P_m \xrightarrow{\varphi} \mathbb{F}_p \widehat{\otimes}_{\mathbb{Z}_p[[H]]} Im(\partial_m) \longrightarrow 0.$$

Note that P_m can be chosen in such a way that the generating set of P_m has the minimal possible cardinality for a fixed $Ker(\partial_{m-1})$ i.e. this is equivalent with φ being an isomorphism. Then $\mathbb{F}_p \widehat{\otimes}_{\mathbb{Z}_p[[H]]} M = 0$ and since $\mathbb{Z}_p[[H]]$ is a local ring, $M = 0$ [7, Theorem 1.5]. $\square$

5.2. *Proof of Corollary 4*

By Lemma 3 any open subgroup of G has homological type FP_n, has cohomological dimension that equals $cd_p(G) = n$ and is of Euler characteristics 0. As the quotient group G/N is of finite rank by the results from the preliminaries, namely Theorem 7, G/N is virtually uniformly powerful. Thus we can assume that G/N is uniformly powerful. Then as G/N is uniform it does not have non-trivial elements of finite order [9, Theorem 4.8].

By the main result of [20] the completed group algebra $\mathbb{Z}_p[[H]]$ of analytic torsion-free pro-p group H does not have zero divisors. If H is uniformly powerful $\mathbb{F}_p[[H]]$ does not have zero divisors [15, Section 4, Theorem 2] and $\mathbb{F}_p[[H]]$ is left and right Noetherian [32, 8.6.5, 8.7.10]. Then if N has infinite index in G we can apply Theorem 4. Thus the group N is of homological type FP_n and G/N has finite virtual cohomological dimension $vcd_p(G/N) = cd_p(G) - cd_p(N)$, in particular $cd_p(N) \leqslant n - 1$. If N has finite index in G then N has the same homological type and cohomological dimension as G.

5.3. *Proof of Corollary 5*

Let

$$\cdots \longrightarrow F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 = \mathbb{Z}_p[[G]] \xrightarrow{\partial_0} \mathbb{Z}_p \longrightarrow 0$$

be a free resolution of the trivial (left) pro-p $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p$ with F_i being of finite rank m_i for $i = 0, 1, 2$ and $m_2 + m_0 = m_1$. This is possible as G has deficiency 1. As in the proof of Proposition 2 we apply the functor $\mathbb{F}_p\widehat{\otimes}_{\mathbb{Z}_p[[N]]}$ to this complex and obtain a complex

$$\begin{aligned}\mathscr{K}:\cdots\longrightarrow \mathbb{F}_p\widehat{\otimes}_{\mathbb{Z}_p[[N]]}F_3 &\xrightarrow{d_3} \mathbb{F}_p[[G/N]]^{m_2} \xrightarrow{d_2} \mathbb{F}_p[[G/N]]^{m_1}\\ &\xrightarrow{d_1} \mathbb{F}_p[[G/N]]^{m_0} \xrightarrow{d_0} \mathbb{F}_p \longrightarrow 0\end{aligned}$$

with $H_1(\mathscr{K})\simeq H_1(N,\mathbb{F}_p)\simeq N/\overline{N^p[N,N]}$ finite and $H_0(\mathscr{K})=0$. Let S be the abstract field of fractions of $\mathbb{F}_p[[G/N]]\simeq\mathbb{F}_p[[t]]$. Applying the abstract tensor product $S\otimes_{\mathbb{F}_p[[G/N]]}$ to $\mathscr{K}$ we obtain a complex of abstract S-modules

$$\begin{aligned}S\otimes_{\mathbb{F}_p[[G/N]]}\mathscr{K}:\cdots\longrightarrow C_3 &\xrightarrow{id\otimes d_3} C_2=S^{m_2}\xrightarrow{id\otimes d_2} C_1=S^{m_1}\\ &\longrightarrow C_0=S^{m_0}\longrightarrow C_{-1}=S\otimes_{\mathbb{F}_p[[G/N]]}\mathbb{F}_p=0\end{aligned}$$

with $H_i(S\otimes_{\mathbb{F}_p[[G/N]]}\mathscr{K})=0$ for $i=0,1$ since $H_i(\mathscr{K})$ is finite for $i=0,1$. Counting dimensions over S we get that $id\otimes d_2$ is injective, hence d_2 is injective and $H_2(N,\mathbb{F}_p)=H_2(\mathscr{K})=Ker\,d_2/Im\,d_3=0$. Thus by Lemma 5 $cd_p(N)\leqslant 1$ and so $cd_p(G')\leqslant cd_p(N)\leqslant 1$. By Serre's result [24, Theorem 7.7.4] either $N=1$, hence $G\simeq\mathbb{Z}_p$ or N is a non-trivial free pro-p group. In the latest case $cd_p(G)\leqslant cd_p(N)+cd_p(G/N)=2$. As N is (topologically) finitely generated by [24, Proposition 7.4.2(b)(i)] we have equality $cd_p(G)=cd_p(N)+cd_p(G/N)=2$.

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References

[1] P. Ara, K.C. O'Meara, F. Perera, Stable finiteness of group rings in arbitrary characteristic, Adv. Math. 170 (2) (2002) 224–238.
[2] R. Bieri, Normal subgroups in duality groups and in groups of cohomological dimension 2, J. Pure Appl. Algebra 7 (1) (1976) 35–51.
[3] R. Bieri, Homological dimension of discrete groups, second ed., Queen Mary College Mathematical Notes. Queen Mary College, Department of Pure Mathematics, London, 1981.
[4] R. Bieri, B. Renz, Valuations on free resolutions and higher geometric invariants of groups, Comment. Math. Helv. 63 (1988) 464–497.
[5] R. Bieri, R. Strebel, Geometric Invariants for Discrete groups, manuscript-book, Frankfurt University.
[6] K.S. Brown, Cohomology of groups, Graduate Texts in Mathematics, vol. 87, Springer, New York, 1994.
[7] A. Brumer, Pseudocompact algebras, profinite groups and class formations, J. Algebra 4 (1966) 442–470.
[8] W. Dicks, M.J. Dunwoody, Groups acting on graphs, Cambridge Studies in Advanced Mathematics, vol. 17, CUP, 1989.
[9] J.D. Dixon, M.P.F. du Sautoy, A. Mann, D. Segal, Analytic pro-p groups, second ed., Cambridge Studies in Advanced Mathematics, vol. 61, Cambridge University Press, Cambridge, 1999.
[10] K.R. Goodearl, R.B. Warfield, An introduction to noncommutative Noetherian rings, London Mathematical Society Student Texts, vol. 16, Cambridge University Press, Cambridge, 1989.

[11] D. Gottlieb, A certain subgroup of the fundamental group, Amer. J. Math. 87 (1965) 840–856.
[12] J.A. Hillman, On L^2-homology and asphericity, Israel J. Math. 99 (1997) 271–283.
[13] J. Hillman, D. Kochloukova, Finiteness conditions and PD_r group covers of PD_n-complexes, in preparation.
[14] I. Kaplansky, Fields and Rings, University of Chicago Press, Chicago, IL, 1969.
[15] J.D. King, Homological finiteness conditions for pro-p groups, Comm. Algebra 27 (10) (1999) 4969–4991.
[16] M. Lazard, Groupes analytiques p-adiques, Inst. Hautes Études Scientifiques, Publ. Math. 26, 389–603.
[17] A. Lubotzky, A. Mann, Powerful p-groups. II. p-adic analytic groups, J. Algebra 105 (2) (1987) 506–515.
[18] H. Meinert, The geometric invariants of direct products of virtually free groups, Commentarii Math. Helv. 69 (1) (1994) 39–48.
[19] M.S. Montgomery, Left and right inverses in group algebras, Bull. Amer. Math. Soc. 75 (1969) 539–540.
[20] A. Neumann, Completed group algebras without zero divisors, Archiv. Math. 51 (6) (1988) 496–499.
[21] D.S. Passman, The Algebraic Structure of Group Rings, Robert Krieger Publishing Company, 1985.
[22] A. Ranicki, Finite domination and Novikov rings, Topology 34 (3) (1995) 619–632.
[23] E. Rapaport Strasser, Knot-like groups, in Knots, groups and 3-manifolds, in: L.P. Neuwirt (Ed.), Annals of Mathematical Study, vol. 84, Princeton University Press, Princeton, NJ, 1975, pp. 119–133.
[24] L. Ribes, P. Zalesskii, Profinite Groups, Springer, Berlin, 2000.
[25] S. Rosset, A vanishing theorem for Euler characteristics, Math. Z. 185 (1984) 211–215.
[26] J. Rotman, An Introduction to Homological Algebra, Academic Press, New York, 1979.
[27] P. Schweizer, Finiteness properties and vanishing of group homology with Novikov-ring coefficients, Diploma Thesis, Frankfurt University, 2005.
[28] J.P. Serre, Cohomologie Galoisienne, Lecture Notes in Mathematics, vol. 5, Springer, Berlin, 1994.
[29] J.P. Sikorav, Homologie de Novikov associee a une classe de cohomologie reelle de degree un, preprint 1989.
[30] P. Symonds, Th. Weigel, Cohomology of p-adic analytic group, New horisons in pro-p groups, in: M. du Sautoy, A. Segal, A. Shalev (Eds.), Progress in Maths, vol. 184, Birkhauser, Basel, 2000.
[31] Th. Weigel, P.A. Zalesskii, Profinite groups of finite cohomological dimension, C. R. Acad. Sci. Paris Ser. I 338 (2004) 353–358.
[32] J.S. Wilson, Profinite Groups, Clarendon Press, Oxford, 1998.